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Bäcklund Transformations for Harmonic Maps in Two Independent Variables

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Bäcklund transformations for harmonic maps are described as the action of the structure group on harmonic one-forms or as gauge transformations of the soliton connection constructed via embedding the configuration manifold into a flat space. As an illustration, Bäcklund transformations for maps from M^2 to the Poincaré upper half-plane and for maps determining stationary vacuum gravitational fields with axial symmetry are obtained.

1. INTRODUCTION

Since its mathematical theory was first established by Eells and Sampson (1964), harmonic maps (HM) have been exploited considerably in physics especially through their applications in nonlinear σ-models (Sanchez, 1982), Yang–Mills fields (Chau *et al.*, 1981), and Einstein's equations (Eriş, 1976; Eriş and Nutku, 1974). Formulation of a physical theory in the framework of HM not only provides a deeper insight into the geometrical background of the theory (Misner, 1978), but also leads to the generation of new solutions to its field equations. In this work, the *energy* functional and its Euler–Lagrange equations are expressed in terms of differential forms, which offers a natural distinction between the kinematical and the dynamical content of the theory, and thus uncovers its Hamiltonian structure. This formulation is also found useful for various descriptions of Bäcklund transformations (BT), which are in general differential equations of order less than that of the original differential equation relating any solution of the differential equation to another solution of the same equation. Usually, specific models have their specific methods for

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generating new solutions. Here these are either attained by the action of the structure group on harmonic one-forms and (or) by gauge transforming the *soliton* connection constructed by embedding the configuration manifold into a flat manifold such that the transformation parameters satisfy the HM equations. As an illustration, first a model field theory for which the configuration space for the fields is the Poincaré upper half-plane (PUP) is demonstrated to be completely integrable. Then the HM formulation of the stationary vacuum gravitational fields with axial symmetry (SVAS) is considered, and its BT are established.

2. HARMONIC MAPS AND BÄCKLUND TRANSFORMATIONS

2.1. Harmonic Maps

Let $f: (M, g) \to (M', g')$ be a smooth map between two Riemannian manifolds of dimensions m and m' with metrics $ds^2 = g_{ab}\, dx^a\, dx^b$ and $ds'^2 = g'_{AB}\, dy^A\, dy^B$, respectively. The trace of the induced metric $f^* \circ g'$ on M is called the energy density, and the energy functional is written as

$$E(f) = \int_M \operatorname{tr}(f^* \circ g')\, {\star}1 = \frac{1}{2} \int_M \eta_{\mu\nu} \sigma^\mu \wedge {\star}\sigma^\nu \tag{1}$$

where ${\star}1$ is the invariant volume element on M. If ω^μ is a basis for the cotangent space T^*_y such that $ds'^2 = \eta_{\mu\nu}\omega^\mu \otimes \omega^\nu$, then the pullback $\sigma^\mu = f^* \circ \omega^\mu$ is a set of one-forms whose values form a vector bundle on M. They satisfy

$$D\sigma^\mu \equiv d\sigma^\mu + \Omega^\mu{}_\nu \wedge \sigma^\nu = 0 \tag{2}$$

where $\Omega^\mu{}_\nu$ is the pulled-back connection one-form of M' on M. With this definition of $\Omega^\mu{}_\nu$, the map f is harmonic if

$$D\, {\star}\sigma^\mu \equiv d{\star}\sigma^\mu + \Omega^\mu{}_\nu \wedge {\star}\sigma^\nu = 0 \tag{3}$$

where ${\star}: T^{s*}(M) \to T^{m-s*}(M)$ is the Hodge dual map defined by the Riemannian structure of M. Consistency conditions of (2) and (3) are, respectively, $\Theta^\mu{}_\nu \wedge \sigma^\nu = 0$ and $\Theta^\mu{}_\nu \wedge {\star}\sigma^\nu = 0$, where $\Theta^\mu{}_\nu = d\Omega^\mu{}_\nu + \Omega^\mu{}_\alpha \wedge \Omega^\alpha{}_\nu$ is the curvature two-form on M induced from M'. In what follows, one-forms σ^μ satisfying (2) and (3) will be referred to as harmonic one-forms.

2.2. Bäcklund Transformations

One can write $\sigma^\mu = P^\mu{}_a\, dx^a$, where $P^\mu{}_a(f^B, f^B{}_{,b}) = h^\mu{}_A(f^B) f^A{}_{,a}$ and $h^\mu{}_A h_{\mu B} = g'_{AB}$; then (3) is a system of first-order differential equations for the unknown functions $P^\mu{}_a(f^B, f^B{}_{,b})$. However this definition of $P^\mu{}_a$ is not unique, since a transformation of the form $P^{\mu}{}_a(f^B, f^B{}_{,b}) \to P'^{\mu}{}_a(\{\alpha\}, f^B, f^B{}_{,b})$ can be found which leaves the field equations (2) and (3) invariant, where $\{\alpha\}$ denotes some set of parameters. An immediate example is the transformation of basis one-forms

$$\sigma^{\mu} \to \sigma'^{\mu} = A^{\mu}{}_{\nu}\sigma^{\nu} \tag{4}$$

such that the action integral (1) is left-form-invariant. This can geometrically be interpreted as the action of the structure group G on the principal fiber $P = (E, \pi, M, G)$, where $\pi: f^*T^*(M') \to M$. If M is a two-dimensional manifold, ${}^{\star}\sigma^{\mu}$ are again one-forms; therefore one can consider the transformation

$$\sigma^{\mu} \to \sigma'^{\mu} = B^{\mu}{}_{\nu}\, {}^{\star}\sigma^{\nu} \tag{5}$$

or more generally the mixed transformation

$$\sigma^{\mu} \to \sigma'^{\mu} = A^{\mu}{}_{\nu}\sigma^{\nu} + B^{\mu}{}_{\nu}\, {}^{\star}\sigma^{\nu} \tag{6}$$

Using such a transformation and given a harmonic map $f^B(x^a)$, one can define a new map $f'^B(x^a)$ through first-order differential equations

$$P^{\mu}{}_a(f'^B, f'^B{}_{,b}) = P'^{\mu}{}_a(\{\alpha\}, f^B, f^B{}_{,b}) \tag{7}$$

with $\{\alpha\} = \{\alpha(x^a)\}$. The integrability of (7) as well as the harmonicity of the map $f'^B(x^a)$ are satisfied if the transformed one-forms $\sigma'^{\mu} = P^{\mu}{}_a(\{\alpha\}, f^B, f^B{}_{,b})\, dx^a$ are harmonic:

$$D\sigma'^{\mu} = 0, \qquad D\, {}^{\star}\sigma'^{\mu} = 0 \tag{8}$$

which are first-order differential equations for the set of parameters α. Since $\sigma^{\mu} = P^{\mu}{}_a\, dx^a$ are harmonic one-forms, these equations are already integrable. Then (7) defines a Bäcklund transformation for the HM equations.

If the induced connection on M can be written as $\Omega^{\mu}{}_{\nu} = \gamma^{\mu}{}_{\nu\alpha}\sigma^{\alpha}$, with $\gamma^{\mu}{}_{\nu\alpha} = \text{const}$, then for σ'^{μ} in (6) to satisfy (8) the differential conditions on A and B are

$$\begin{aligned} dA^{\mu}{}_{\nu} - \gamma^{\lambda}{}_{\nu\beta}A^{\mu}{}_{\lambda}\sigma^{\beta} + \gamma^{\mu}{}_{\lambda\rho}A^{\rho}{}_{\beta}A^{\lambda}{}_{\nu}\sigma^{\beta} + \gamma^{\mu}{}_{\lambda\rho}B^{\rho}{}_{\beta}A^{\lambda}{}_{\nu}\, {}^{\star}\sigma^{\beta} = 0 \\ dB^{\mu}{}_{\nu} - \gamma^{\lambda}{}_{\nu\beta}B^{\mu}{}_{\lambda}\sigma^{\beta} + \gamma^{\mu}{}_{\lambda\rho}B^{\rho}{}_{\beta}B^{\lambda}{}_{\nu}\, {}^{\star}\sigma^{\beta} + \gamma^{\mu}{}_{\lambda\rho}A^{\rho}{}_{\beta}B^{\lambda}{}_{\nu}\sigma^{\beta} = 0 \end{aligned} \tag{9}$$

The integrability of these equations yields the following conditions:

$$\begin{aligned} -M^{\lambda}{}_{\nu\alpha\beta}A^{\mu}{}_{\lambda} + M^{\mu}{}_{\lambda\rho\sigma}A^{\lambda}{}_{\nu}A^{\rho}{}_{\beta}A^{\sigma}{}_{\alpha} + M^{\mu}{}_{\lambda\rho\sigma}A^{\lambda}{}_{\mu}B^{\rho}{}_{\alpha}B^{\sigma}{}_{\beta} = 0 \\ M^{\mu}{}_{\lambda\rho\sigma}A^{\lambda}{}_{\nu}B^{\rho}{}_{\alpha}A^{\sigma}{}_{\beta} = 0 \\ -M^{\lambda}{}_{\nu\alpha\beta}B^{\mu}{}_{\lambda} + M^{\mu}{}_{\lambda\rho\sigma}B^{\lambda}{}_{\nu}B^{\rho}{}_{\beta}B^{\sigma}{}_{\alpha} + M^{\mu}{}_{\lambda\rho\sigma}B^{\lambda}{}_{\mu}A^{\rho}{}_{\alpha}A^{\sigma}{}_{\beta} = 0 \\ M^{\mu}{}_{\lambda\rho\sigma}B^{\lambda}{}_{\nu}A^{\rho}{}_{\alpha}B^{\sigma}{}_{\beta} = 0 \end{aligned} \tag{10}$$

Here,

$$M^{\mu}{}_{\nu\alpha\beta} = (\gamma^{\mu}{}_{\lambda\alpha}\gamma^{\lambda}{}_{\nu\beta} + \gamma^{\mu}{}_{\nu\lambda}\gamma^{\lambda}{}_{\alpha\beta}) \tag{11}$$

and $M^{\mu}{}_{\nu[\alpha\beta]} = \frac{1}{2}R^{\mu}{}_{\nu\alpha\beta}$, where the brackets denote antisymmetrization and $R^{\mu}{}_{\nu\alpha\beta}$ is the Riemann tensor, which is defined by $\Theta^{\mu}{}_{\nu} = \frac{1}{2}R^{\mu}{}_{\nu\alpha\beta}\sigma^{\alpha} \wedge \sigma^{\beta}$. If one considers the transformations (4) and (5) separately, then to obtain the conditions on A and B, it is sufficient to drop the last terms in equations (9) and (10).

2.3. Flat Connection

Due to the curvature of M', HM equations are highly nonlinear second-order differential equations, for which it is not only difficult to find solutions, but also not at all obvious whether they are integrable or not. Instead of dealing with the HM equations, a system of algebraic and first-order differential equations which is linear with respect to the first derivatives can be set up, such that the existence of their solutions implies the integrability of the HM equations. This procedure consists in the construction of a flat connection by embedding M' into a flat manifold. A Riemannian manifold of dimension m' can always be embedded into a flat manifold V_n of dimension $n = m' + p \leq \frac{1}{2}m'(m' + 1)$, where p is the class of M' (Eisenhart, 1966), whereas for the equations under consideration to be integrable, it is required that the curvature two-form of V_n should not vanish identically, but with reference to the field equations. Since the property of being flat is invariant under gauge transformations, once such a flat *soliton* connection Ω is constructed, BT follow readily. If

$$\Omega \to \Omega' = A^{-1}\, dA + A^{-1}\Omega A \tag{12}$$

then the curvature two-form transforms according to $\Theta \to \Theta' = A^{-1}\Theta A$, where A is an element of the group G associated with the Lie algebra, which is determined by the Riemannian structures of M, M', and V_n. Hence, BT are nothing but the gauge transformations, which leave the field equations form-invariant (Crampin, 1978).

We introduce the connection of V_n in the form

$$\Omega^{R}{}_{S} \equiv \begin{pmatrix} \Omega^{\mu}{}_{\nu} & \Pi^{\mu}{}_{j} \\ \Pi^{i}{}_{\nu} & S^{i}{}_{j} \end{pmatrix} \tag{13}$$

with the range of indices $\mu, \nu = 1, \ldots, m'$, $i, j = m' + 1, \ldots, n$, and $R, S = 1, \ldots, n$, while flatness requires

$$D\Omega^{R}{}_{S} \equiv d\Omega^{R}{}_{S} + \Omega^{R}{}_{Q} \wedge \Omega^{Q}{}_{S} = 0 \tag{14}$$

Here $\Omega^{\mu}{}_{\nu}$ is the pulled-back connection of M' on M. The $\Pi^{\mu}{}_{j}$ and $S^{i}{}_{j}$ are subject to the equations

$$\begin{aligned} \Theta^{\mu}{}_{\nu} &= -\Pi^{\mu}{}_{i} \wedge \Pi^{i}{}_{\nu} \\ d\Pi^{\mu}{}_{i} + \Omega^{\mu}{}_{\alpha} \wedge \Pi^{\alpha}{}_{i} &= -\Pi^{\mu}{}_{k} \wedge S^{k}{}_{i} \\ dS^{i}{}_{j} + S^{i}{}_{k} \wedge S^{k}{}_{j} &= -\Pi^{i}{}_{\nu} \wedge \Pi^{\nu}{}_{j} \end{aligned} \tag{15}$$

where $\Theta^{\mu}{}_{\nu}$ is by construction the curvature of M' on M. Any two-form $-\Pi^{\mu}{}_{i} \wedge \Pi^{i}{}_{\nu}$, where $\Pi^{\mu}{}_{i}$ is the solution of the differential equations in (15), satisfies the Bianchi identity. Defining $\Pi^{\mu}{}_{i}$ as $\Pi^{\mu}{}_{i} = Q^{\mu}{}_{i\nu} \star\sigma^{\nu}$ with dimension $m = 2$, we find that the first equation in (15) reduces to

$$R^{\mu}{}_{\nu\alpha\beta} = e\eta_{\nu\rho}\eta^{ki}(Q^{\mu}{}_{k\alpha}Q^{\rho}{}_{i\beta} - Q^{\mu}{}_{k\beta}Q^{\rho}{}_{i\alpha}) \tag{16}$$

where $Q^{\mu}{}_{i\alpha}$ are unknown functions of the coordinates of M' and e is determined by the signature of M. It follows that, if M' is of constant curvature, then $Q^{\mu}{}_{i\alpha}$ can be chosen as constants.

Restricting the dimensions to $m = m' = 2$, $n = 3$, and letting the metric of V_3 be $ds^2 = \mathrm{diag}(1, \epsilon, \eta)$, $\epsilon = \pm 1$, $\eta = \pm 1$, we denote

$$\Pi^{1}{}_{3} \equiv \Pi^{1} = p \star\sigma^{1} + q \star\sigma^{2}, \qquad \Pi^{2}{}_{3} \equiv \Pi^{2} = r \star\sigma^{1} + s \star\sigma^{2} \tag{17}$$

In this case $(S^{i}{}_{j}) = (0)$ and w is the only component of the induced connection on M. Field equations (2) and (3) are explicitly written as

$$\begin{aligned} &d\sigma^{1} + \omega \wedge \sigma^{2} = 0, \qquad d\sigma^{2} - \epsilon\omega \wedge \sigma^{1} = 0 \\ &d \star\sigma^{1} + \epsilon\omega \wedge \star\sigma^{2} = 0, \qquad d \star\sigma^{2} - \epsilon\omega \wedge \star\sigma^{1} = 0 \end{aligned} \tag{18}$$

In view of (18) and using (17), flatness requires

$$\begin{aligned} d\omega - e\epsilon\eta(ps - qr)\sigma^{1} \wedge \sigma^{2} &= 0 \\ (dp + (\epsilon q + r)\omega) \wedge \star\sigma^{1} + (dq + (s - p)\omega) \wedge \star\sigma^{2} &= 0 \\ (dr + \epsilon(s - p)\omega) \wedge \star\sigma^{1} + (ds - (\epsilon q + r)\omega) \wedge \star\sigma^{2} &= 0 \end{aligned} \tag{19}$$

and hence

$$K = e\epsilon\eta(ps - qr) \tag{20}$$

is the Gaussian curvature of M'. The functions p, q, r, s are the solutions of the particular set of equations

$$\begin{aligned} &dp + (\epsilon q + r)\omega = a\sigma^{2}, \qquad dq + (s - p)\omega = -a\sigma^{1} \\ &dr + \epsilon(s - p)\omega = b\sigma^{2}, \qquad ds - (\epsilon q + r)\omega = -b\sigma^{1} \end{aligned} \tag{21}$$

for arbitrary functions a and b. These equations are integrable if

$$(da + b\omega - K(\epsilon q + r)\sigma^1) \wedge \sigma^2 = 0, \qquad (da + b\omega - K(s - p)\sigma^2) \wedge \sigma^1 = 0$$
$$(db - \epsilon a\omega - K\epsilon(s - p)\sigma^1) \wedge \sigma^2 = 0, \qquad (db - \epsilon a\omega + K(\epsilon q + r)\sigma^2) \wedge \sigma^1 = 0 \tag{22}$$

3. EXAMPLES

3.1. The Poincaré Upper Half-Plane

Consider two Riemannian manifolds M and M' with metrics

$$ds^2 = dx^2 - dy^2 \tag{23}$$

and

$$ds'^2 = \xi^{-2}(d\xi^2 + d\eta^2) \tag{24}$$

The manifold M' is called the Poincaré upper half-plane, which is the configuration space for the fields $\xi(x, y)$, $\eta(x, y)$. The tetrad on M' is chosen to be

$$\omega^1 = \xi^{-1}\, d\xi, \qquad \omega^2 = \xi^{-1}\, d\eta \tag{25}$$

and they have their pullbacks on M as

$$\sigma^1 = \xi^{-1}(\xi_x\, dx + \xi_y\, dy), \qquad \sigma^2 = \xi^{-1}(\eta_x\, dx + \eta_y\, dy) \tag{26}$$

where σ^2 is the only component of the induced connection on M with coefficients

$$\begin{aligned} &\gamma^1{}_{11} = \gamma^2{}_{12} = \gamma^2{}_{21} = \gamma^2{}_{22} = \gamma^2{}_{11} = \gamma^1{}_{21} = 0 \\ &\gamma^1{}_{22} = -\gamma^2{}_{12} = 1. \end{aligned} \tag{27}$$

With respect to the Riemannian structure of M, the Hodge dual is defined as $^\star dx = dy$ and $^\star dy = dx$. As $\omega = \sigma^2$ and $\epsilon = +1$, HM equations (18) can easily be rewritten. To obtain a BT as in (6), denote the transformation matrices as

$$A = \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix}, \qquad B = \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix} \tag{28}$$

Then (9) and (10) have solutions

$$\begin{aligned} s_1 = p_1, \qquad r_1 = -q_1, \qquad & s_2 = p_2, \qquad r_2 = -q_2, \\ p_1 = \cos\lambda \cos\theta, \qquad & q_1 = \cos\lambda \sin\theta \\ p_2 = \sin\lambda \cos\theta, \qquad & q_2 = \sin\lambda \sin\theta \end{aligned} \tag{29}$$

where $\lambda = \text{const}$ and

$$d\theta = -\cos\lambda \sin\theta\, \sigma^1 + (\cos\lambda \cos\theta - 1)\sigma^2$$
$$- \sin\lambda \sin\theta\, {}^\star\sigma^1 + \sin\lambda \cos\theta\, {}^\star\sigma^2 \qquad (30)$$

Note that BT relative to (4) or (5) can be obtained simply by setting $\lambda = 0$ or $\lambda = \pi/2$, respectively.

To investigate the integrability of the HM equations, it is of convenience to go to the null coordinates $2x = u + v$ and $2y = u - v$; hence the Hodge dual is redefined as ${}^\star du = du$ and ${}^\star dv = -dv$. This fixes the indicator $e = +1$. From $d\omega = -\omega^1 \wedge \omega^2$ it is seen that the Gaussian curvature of M' is -1. With the choice of $\eta = +1$ and considering (20) one possible parametrization is

$$p = s = \cos\lambda, \qquad q = -r = \sin\lambda \qquad (31)$$

For simplicity the arbitrary functions a and b are set to zero. Then (21) or (22) implies $\lambda = \text{const}$. With this choice of η, Ω is an $so(3)$-valued connection one-form. It is also possible to choose $\eta = -1$, which then forces changes in the parametrization of (20), hence resulting an $su(2, 1)$-valued connection one-form. Taking

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad (32)$$

as the infinitesimal 3×3 matrix generators of $SO(3)$, we can express the connection Ω as

$$\Omega = \omega X_1 - \Pi^1 X_2 + \Pi^2 X_3 \qquad (33)$$

One can also note that, defining new generators x_i by

$$x_1 = -2iX_1, \qquad x_2 = X_2 - iX_3, \qquad x_3 = -(X_2 + iX_3) \qquad (34)$$

which satisfy the commutation relations of $sl(2, R)$, we can construct an AKNS connection (Ablowitz *et al.*, 1973)

$$\Omega = \frac{i}{2}\omega x_1 + \frac{1}{2}(i\Pi^2 - \Pi^1)x_2 + \frac{1}{2}(i\Pi^2 + \Pi^1)x_3 \qquad (35)$$

by making use of the 2×2 infinitesimal matrix generators of $SL(2, R)$.

One-parameter subgroups of $SO(3)$ associated with (32) are

$$A(\theta_1)=\begin{pmatrix}\cos\theta_1 & -\sin\theta_1 & 0\\ \sin\theta_1 & \cos\theta_1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$A(\theta_3)=\begin{pmatrix}\cos\theta_2 & 0 & -\sin\theta_2\\ 0 & 1 & 0\\ \sin\theta_2 & 0 & \cos\theta_2\end{pmatrix} \tag{36}$$

$$A(\theta_3)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta_3 & \sin\theta_3\\ 0 & -\sin\theta_3 & \cos\theta_3\end{pmatrix}$$

Then, by their action on Ω as in (12), BT are found as

$$\begin{aligned}\sigma'^1 &= \cos\theta_1\,\sigma^1 - \sin\theta_1\,\sigma^2\\ \sigma'^2 &= \sin\theta_1\,\sigma^1 + \cos\theta_1\,\sigma^2\end{aligned} \tag{37}$$

and

$$\begin{aligned}\sigma'^1 &= \cos\theta_2\ \sigma^1 + \sin\theta_2\cos\lambda\ {}^\star\sigma^1 - \sin\theta_2(\cos\lambda\cot\lambda + \operatorname{cosec}\lambda)\ {}^\star\sigma^2\\ \sigma'^2 &= \cos\theta_2\,\sigma^2 + \sin\theta_2\sin\lambda\ {}^\star\sigma^1 - \sin\theta_2\cos\lambda\ {}^\star\sigma^2\end{aligned} \tag{38}$$

where the transformation parameters are subject to the equations

$$\begin{aligned}d\theta_1 &= \sin\theta_1\,\sigma^1 + (\cos\theta_1 - 1)\sigma^2\\ d\theta_2 &= -\sin\theta_2\,\sigma^1 + \sin\theta_2(\cos^2\lambda\cot\lambda + \cot\lambda + \sin\lambda\cos\lambda)\sigma^2\\ &\quad - \cos\lambda(\cos\theta_2 - 1)\ {}^\star\sigma^1 - \sin\lambda(\cos\theta_2 - 1)\ {}^\star\sigma^2\end{aligned} \tag{39}$$

With the replacement $\lambda \to \lambda + \pi/2$ and $\theta_2 \to -\theta_3$, BT relative to $A(\theta_3)$ is equivalent to the BT relative to $A(\theta_2)$. Also note that with $\lambda = 0$ and $\theta \to \theta_1$, transformation (6) with (29) and (30) is equivalent to the BT relative to $A(\theta_1)$. Since BT are noncommutative, successive applications can be performed to generate an infinite family of solutions depending on one constant and two variable parameters.

3.2. Stationary Axisymmetric Gravitational Fields

As another example, we consider the problem of determining stationary vacuum gravitational fields with axial symmetry. Following the approach due to Ernst (1968), relevant field equations of the theory are derivable from the action integral

$$I = \int_0^\infty d\rho \int_{-\infty}^{+\infty} dz\{-\nabla\gamma \cdot \nabla\lambda + \lambda\xi^{-2}((\nabla\xi)^2 + (\nabla\phi)^2)\} \tag{40}$$

where ∇ denotes the flat-space grad operator in coordinates ρ, z. (The more conventional f is replaced by ξ so that the complex Ernst potential reads $\epsilon = \xi + i\phi$.) The form of the action integral allows us to formulate the problem in terms of harmonic maps f: $M \to M'$, where M and M' are two Riemannian manifolds with metrics

$$ds^2 = d\rho^2 + dz^2 \tag{41}$$

and

$$ds'^2 = -d\gamma\, d\lambda + \lambda\xi^{-2}(d\xi^2 + d\phi^2) \tag{42}$$

respectively. Choosing the tetrad on M' to be

$$\omega^0 = d\gamma, \qquad \omega^1 = d\lambda, \qquad \omega^2 = \lambda^{1/2}\xi^{-1}\, d\xi, \qquad \omega^3 = \lambda^{1/2}\xi^{-1}\, d\phi \tag{43}$$

we find the nonvanishing components of the connection as

$$\omega^2{}_1 = \frac{1}{2}\,\omega^0{}_2 = \frac{1}{2}\,\lambda^{-1}\omega^2, \quad \omega^3{}_1 = \frac{1}{2}\,\omega^0{}_3 = \frac{1}{2}\,\lambda^{-1}\omega^3, \quad \omega^2{}_3 = -\omega^3{}_2 = \lambda^{-1/2}\omega^3 \tag{44}$$

Their pullbacks on M are the set of one-forms

$$\begin{aligned} \sigma^0 &= \gamma_\rho\, d\rho + \gamma_z\, dz, \qquad & \sigma^2 &= \lambda^{1/2}\xi^{-1}\xi_\rho\, d\rho + \lambda^{1/2}\xi^{-1}\xi_z\, dz \\ \sigma^1 &= \lambda_\rho\, d\rho + \lambda_z\, dz, \qquad & \sigma^3 &= \lambda^{1/2}\xi^{-1}\phi_\rho\, d\rho + \lambda^{1/2}\xi^{-1}\phi_z\, dz \end{aligned} \tag{45}$$

whose values form a four-dimensional Riemannian-connected vector bundle over M. Defining the Hodge dual operation as ${}^\star d\rho = dz$, ${}^\star dz = -d\rho$, we find that the action integral (40) takes the form

$$I = \int_M (-\sigma^0 \wedge {}^\star\sigma^1 + \sigma^2 \wedge {}^\star\sigma^2 + \sigma^3 \wedge {}^\star\sigma^3) \tag{46}$$

while three of the field equations read

$$\begin{aligned} D\, {}^\star\sigma^1 &\equiv d\, {}^\star\sigma^1 = 0 \\ D\, {}^\star\sigma^2 &\equiv d\, {}^\star\sigma^2 + \tfrac{1}{2}\lambda^{-1}\sigma^2 \wedge {}^\star\sigma^1 + \lambda^{+1/2}\sigma^3 \wedge {}^\star\sigma^3 = 0 \\ D\, {}^\star\sigma^3 &\equiv d\, {}^\star\sigma^3 + \tfrac{1}{2}\lambda^{-1}\sigma^3 \wedge {}^\star\sigma^1 - \lambda^{-1/2}\sigma^3 \wedge {}^\star\sigma^2 = 0 \end{aligned} \tag{47}$$

The remaining equation $D\, {}^\star\sigma^0 = 0$ is omitted simply because once a solution to (47) is found, γ can be determined by quadratures. By the redefinition of the basis one-forms

$$\Pi^1 = \lambda^{-1}\sigma^1, \qquad \Pi^2 = \lambda^{-1/2}\sigma^2, \qquad \Pi^3 = \lambda^{-1/2}\sigma^3 \tag{48}$$

the integrability conditions become

$$d\Pi^1 = 0, \qquad d\Pi^2 = 0, \qquad d\Pi^3 - \Pi^3 \wedge \Pi^2 = 0 \tag{49}$$

and the field equations reduce to

$$\begin{aligned} d\,{\star}\Pi^1 + \Pi^1 \wedge {\star}\Pi^1 &= 0 \\ d\,{\star}\Pi^2 + \Pi^2 \wedge {\star}\Pi^1 + \Pi^3 \wedge {\star}\Pi^3 &= 0 \\ d\,{\star}\Pi^3 + \Pi^3 \wedge {\star}\Pi^1 - \Pi^3 \wedge {\star}\Pi^2 &= 0 \end{aligned} \tag{50}$$

Consider the transformation (4), which in this particular example is expressed as

$$\Pi'^i = A^i{}_j \Pi^j + B^i{}_j\, {\star}\Pi^j \qquad (i, j = 1, 2, 3) \tag{51}$$

where

$$(A^i{}_j) = \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\psi\cos\theta & -\sin\psi\cos\theta \\ 0 & \sin\psi\cos\theta & \cos\psi\cos\theta \end{pmatrix} \tag{52}$$

$$(B^i{}_j) = \begin{pmatrix} \sin\beta & 0 & 0 \\ 0 & \cos\psi\sin\theta & -\sin\psi\sin\theta \\ 0 & \sin\psi\sin\theta & \cos\psi\sin\theta \end{pmatrix} \tag{53}$$

These relations define a BT for the HM equations (50) if the set of one-forms Π'^i and Π'^i satisfy equations (49) and (50). Then the parameters $\{\beta, \theta, \psi\}$ are subject to the equations

$$\begin{aligned} d\beta - \sin\beta\,\Pi^1 + (\cos\beta - 1)\,{\star}\Pi^1 &= 0 \\ d\psi - \cos\theta\sin\psi\,\Pi^2 - \sin\theta\sin\psi\,{\star}\Pi^2 & \\ + (1 - \cos\theta\cos\psi)\Pi^3 - \sin\theta\cos\psi\,{\star}\Pi^3 &= 0 \\ \beta &= -2\theta \end{aligned} \tag{54}$$

which are integrable in view of (49) and (50). The field equations (50) and the BT thus obtained are equivalent to those found by Neugebauer (1979), which can be verified in view of the system of first-order differential equations for the unknown functions

$$\begin{aligned} M_1 &= \tfrac{1}{4}\xi^{-1}\{(\xi_\rho + \phi_z - i(\xi_z - \phi_\rho)\}, \qquad & M_3 &= \tfrac{1}{2}\lambda^{-1}(\lambda_\rho - i\lambda z) \\ M_2 &= \tfrac{1}{4}\xi^{-1}\{\xi_\rho - \phi_z - i(\xi_z + \phi_\rho)\}, \qquad & N_k &= M_k \end{aligned} \tag{55}$$

and redefining the parameters θ and ψ as

$$\cos\theta = \frac{1}{2}(\gamma^{1/2} + \gamma^{-1/2}), \qquad \cos\psi = \frac{1}{2}(\alpha\gamma^{-1/2} + \alpha^{-1}\gamma^{1/2}) \tag{56}$$

4. CONCLUSION

We considered some methods to obtain Bäcklund transformations of harmonic maps and applied these to maps which are from M^2 to the Poincaré upper half-plane and from E^2 to a four-dimensional Riemannian manifold. The former example illustrates a model field, where the range manifold is a space of negative constant curvature. In the latter example, using the Ernst potentials, equations for stationary vacuum gravitational fields with axial symmetry are written in terms of harmonic maps, where the configuration manifold is not of constant curvature. Transforming the harmonic one-forms as prescribed in Section 2.2, we find Bäcklund transformations which are equivalent to that of Neugebauer. Recently, construction of Bäcklund transformations based on the Lax-pair approach of the $U(N)$ σ-models has been discussed (Uhlenbeck, 1989). The references therein consider many techniques for finding new solutions of nonlinear field equations. It is known that there is a certain relationship between the symmetric space properties of M' and the σ-model formulation of the field theory expressed in the framework of harmonic maps (Eriş *et al.*, 1984). However, since the embedding procedure is carried out irrespective of the isometries of the configuration manifold, finding Bäcklund transformations to field equations by gauge transforming the soliton connection can be applied to more general problems on the condition that the domain manifold M is effectively two-dimensional.

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REFERENCES

Ablowitz, M. J., Kaup, D. J., Newell, A. C., and Segur, H. (1973). *Physical Review Letters*, **31**, 125.

Chau, L. L., Prasad, M., and Sinha, A. (1981). *Physical Review D*, **24**, 1574.

Crampin, M. (1978). *Physics Letters*, **66A**, 170.

Eells, J., Jr., and Sampson, J. H. (1964). *American Journal of Mathematics*, **86**, 109.

Eisenhart, L. P. (1966). *Riemannian Geometry*, Princeton University Press, Princeton, New Jersey.

Eriş, A. (1976). *Journal of Mathematical Physics*, **18**, 824.
Eriş, A., and Nutku, Y. (1974). *Journal of Mathematical Physics*, **16**, 1431.
Eriş, A., Gürses, M., and Karasu, A. (1984). *Journal of Mathematical Physics*, **25**, 1489.
Ernst, F. J. (1968). *Physical Review*, **167**, 1175.
Misner, C. W. (1978). *Physical Review D*, **18**, 4510.
Neugebauer, G. (1979). *Journal of Physics A*, **12**, L67.
Sanchez, N. (1982). *Physical Review D*, **26**, 2589.
Uhlenbeck, K. (1989). *Journal of Differential Geometry*, **30**, 1.